NUMBERS IN A LATTICE VALUED SET THEORY(New Aspects in Non-Classical Logics and Their Kripke Semantics)

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NUMBERS IN A LATTICE VALUED SET THEORY

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ABSTRACT. Lattice valued set theory **LZFZ** was formulated in [8] as a set theory on a lattice valued universe $V^{\mathcal{L}}$, where we introduced the basic implication $\to$ which represents the order relation on the lattice $\mathcal{L}$.

In this paper, we first prove that the ordinals, natural numbers, rational numbers, and real numbers defined in **LZFZ** are all check sets. Then we add to **LZFZ** an axiom "$\mathcal{P}(1)$ is a cHa". The axiom asserts that the logic is distributive, and enables us to define the intuitionistic implication $\to_{\mathrm{I}}$. Thus, **LZFZ** +"$\mathcal{P}(1)$ is a cHa" is a global intuitionistic set theory which is equivalent to **GIZFZ** in [7]. In the set theory **LZFZ** +"$\mathcal{P}(1)$ is a cHa", the sheaf structures of sets is represented as the relation between the two equalities $=$ and $=_{\mathrm{I}}$ corresponding to two implications $\to$ and $\to_{\mathrm{I}}$, respectively.

INTRODUCTION

In [8], we formulated a set theory which is valid on a lattice valued universe $V^{\mathcal{L}}$, by introducing the basic implication. The basic implication $\to$ on a lattice $\mathcal{L}$ is defined by

$$(a \to b) = \begin{cases} 1 & \text{if } a \leqslant b \\ 0 & \text{otherwise.} \end{cases}$$

The negation $\neg$ corresponding to the basic implication $\to$ is defined by

$$\neg a = (a \to 0).$$

Generally, an operator $\to_*$ on a lattice is called an *implication* if it satisfies the following conditions :

(1) $(a \to_* b) = 1$ if $a \leqslant b$
(2) $a \wedge (a \to_* b) \leqslant b$.

The basic implication is the strongest implication, in the sense that

$$(a \to b) \leqslant (a \to_* b) \quad \text{for any implication } \to_*,$$

and represents the order relation $\leqslant$ on the lattice.

If $\mathcal{L}$ is a complete Heyting algebra(cHa), hence $\mathcal{L}$ is distributive, then we can define another implication $\to_{\mathrm{I}}$ on $\mathcal{L}$ by

$$(a \to_{\mathrm{I}} b) = \bigvee\{c \in \mathcal{L} \mid a \wedge c \leqslant b\}.$$

$\to_{\mathrm{I}}$ is an interpretation of intuitionistic implication.

The *lattice valued set theory* (**LZFZ**) is a set theory which is valid on any lattice valued universe $V^{\mathcal{L}}$ with the basic implication $\to$ and the corresponding negation $\neg$, as well as lattice operators $\wedge$, $\vee$, $\forall$ and $\exists$. The truth value set $\mathcal{L}$ of the set theory **LZFZ** on $V^{\mathcal{L}}$ is represented by the power set $\mathcal{P}(1)$ of $1 = \{0\}$ in the set theory.

We first define the sets of natural numbers, rational numbers, and real numbers, and also ordinals in the set theory **LZFZ** , and prove that they are all check sets. This means that the sets of natural numbers, rational numbers, and real numbers, and ordinals defined in the set theory **LZFZ** have the same properties as those defined in our metamathematics **ZFC** .

Then we add to **LZFZ** an axiom "$\mathcal{P}(1)$ is a cHa" to have a global intuitionistic set theory. That is, **LZFZ** +"$\mathcal{P}(1)$ is a cHa" is a set theory on Heyting valued universe, where the intuitionistic implication $\to_{\mathrm{I}}$ is defined by

$$\varphi \to_{\mathrm{I}} \psi \iff \exists x \in 1(\varphi \wedge (x \in 1) \to \psi)$$

The corresponding equality $=_{\mathrm{I}}$ and the membership relation $\in_{\mathrm{I}}$ can be defined in the set theory so that

$$u =_{\mathrm{I}} v \iff \forall x(x \in_{\mathrm{I}} v \leftrightarrow_{\mathrm{I}} x \in_{\mathrm{I}} v) \ ; \quad u \in_{\mathrm{I}} v \iff \exists x(u =_{\mathrm{I}} x \wedge x \in_{\mathrm{I}} v).$$

The sets of intuitionistic natural numbers and intuitionistic rational numbers defined in the global intuitionistic set theory are check sets, and coincide with those defined in **LZFZ**. But the set of intuitionistic real numbers is not a check set. It has the structure of sheaf, which is written in the language of our set theory **LZFZ** +"$\mathcal{P}(1)$ is a cHa".

1. Preliminary

In this section we review [8] (Lattice valued set theory).

1.1. Lattice valued universe $V^{\mathcal{L}}$. Let $\mathcal{L}$ be a complete lattice with the basic implication $\rightarrow$ and the corresponding negation $\neg$, where the least upper bound of a subset $\{a_\alpha\}_\alpha$ of $\mathcal{L}$ is denoted by $\bigvee_\alpha a_\alpha$, and the greatest lower bound of $\{a_\alpha\}_\alpha$ is denoted by $\bigwedge_\alpha a_\alpha$; the smallest element and the largest element of $\mathcal{L}$ are denoted by 0 and 1, respectively; and the basic implication $\rightarrow$ is the operator on $\mathcal{L}$ defined by

$$(a \rightarrow b) = \begin{cases} 1 & \text{if } a \leqslant b \\ 0 & \text{otherwise.} \end{cases}$$

The complement corresponding to $\rightarrow$ is defined by

$$\neg a = (a \rightarrow 0).$$

We denote the formula $(1 \rightarrow a)$ by $\Box a$, that is,

$$\Box a = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{if } a \neq 1. \end{cases}$$

$\mathcal{L}$-valued universe $V^{\mathcal{L}}$ is constructed by induction:

$$\begin{aligned} V^{\mathcal{L}}_\alpha &= \{u \mid \exists \beta < \alpha \exists \mathcal{D}u \subset V^{\mathcal{Q}}_\beta (u : \mathcal{D}u \rightarrow \mathcal{L})\} \\ V^{\mathcal{L}} &= \bigcup_{\alpha \in \mathrm{On}} V^{\mathcal{L}}_\alpha \end{aligned}$$

The least α such that $u \in V^{\mathcal{L}}_\alpha$ is called the *rank* of u. For $u, v \in V^{\mathcal{L}}$, the truth values $[\![u = v]\!]$ and $[\![u \in v]\!]$ of the atomic sentences $u = v$ and $u \in v$ are defined by induction on the rank of u, v.

$$\begin{aligned} [\![u = v]\!] &= \bigwedge_{x \in \mathcal{D}u} (u(x) \rightarrow [\![x \in v]\!]) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \rightarrow [\![x \in u]\!]) \\ [\![u \in v]\!] &= \bigvee_{x \in \mathcal{D}v} [\![u = x]\!] \wedge v(x). \end{aligned}$$

We say an element p of $\mathcal{L}$ is *global* if $p = \Box p$. It is obvious from the definition that $[\![u = v]\!]$ is global.

1.2. Lattice valued set theory. Now we formulate a set theory on $V^{\mathcal{L}}$, and call it *lattice valued set theory* **LZFZ** .

Atomic symbols of **LZFZ** are:

(1) variables $x, y, z, \cdots$
(2) predicate constants $=, \in$
(3) logical symbols $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\forall$, $\exists$
(4) parentheses (,).

Formulas of **LZFZ** are constructed from atomic formulas of the form $x=y$ or $x\in y$ by using the logical symbols.

We denote a sentence $(\varphi \to \varphi) \to \varphi$ by $\Box\varphi$.

1.2.1. *Lattice valued logic.* *Lattice valued logic*, shortly **L**, is a formalization of the logic on $\mathcal{L}$-valued universe $V^{\mathcal{L}}$. The rules of **L** are given by restricting **LK**. First we define $\Box$-*closed formulas* inductively by :

(1) A formula of the form $\varphi \to \psi$ or $\neg\varphi$ is $\Box$-closed.
(2) If formulas φ and ψ are $\Box$-closed, then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are $\Box$-closed.
(3) If a formula $\varphi(x)$ is a $\Box$-closed formula with free variable x, then $\forall x\varphi(x)$ and $\exists x\varphi(x)$ are $\Box$-closed.
(4) $\Box$-closed formulas are only those obtained by (1)–(4).

$\varphi, \psi, \xi, \cdots, \varphi(x), \cdots$ are used to denote formulas ; $\Gamma, \Delta, \Pi, \Lambda, \cdots$ to denote finite sequences of formulas ; $\overline{\varphi}, \overline{\psi}, \cdots$ to denote $\Box$-closed formulas ; and $\overline{\Gamma}, \overline{\Delta}, \overline{\Pi}, \overline{\Lambda}, \cdots$ to denote finite sequences of $\Box$-closed formulas. A formal expression of the form $\Gamma \Longrightarrow \Delta$ is called a *sequent.*

Logical axioms : Axioms of **L** are sequents of the form $\varphi \Longrightarrow \varphi$.

Structural rules:

Thinning :
$$\frac{\Gamma \Longrightarrow \Delta}{\varphi, \Gamma \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, \varphi}$$

Contraction :
$$\frac{\varphi, \varphi, \Gamma \Longrightarrow \Delta}{\varphi, \Gamma \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta, \varphi, \varphi}{\Gamma \Longrightarrow \Delta, \varphi}$$

Interchange :
$$\frac{\Gamma, \varphi, \psi, \Pi \Longrightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Longrightarrow \Delta, \psi, \varphi, \Lambda}$$

Cut :
$$\frac{\Gamma \Longrightarrow \overline{\Delta}, \varphi \quad \varphi, \Pi \Longrightarrow \Lambda}{\Gamma, \Pi \Longrightarrow \overline{\Delta}, \Lambda} \qquad \frac{\Gamma \Longrightarrow \Delta, \varphi \quad \varphi, \overline{\Pi} \Longrightarrow \Lambda}{\Gamma, \overline{\Pi} \Longrightarrow \Delta, \Lambda}$$

$$\frac{\Gamma \Longrightarrow \Delta, \overline{\varphi} \quad \overline{\varphi}, \Pi \Longrightarrow \Lambda}{\Gamma, \Pi \Longrightarrow \Delta, \Lambda}$$

Logical rules:

$\neg$:
$$\frac{\Gamma \Longrightarrow \overline{\Delta}, \varphi}{\neg\varphi, \Gamma \Longrightarrow \overline{\Delta}} \quad \frac{\Gamma \Longrightarrow \Delta, \overline{\varphi}}{\neg\overline{\varphi}, \Gamma \Longrightarrow \Delta} \qquad \frac{\varphi, \overline{\Gamma} \Longrightarrow \overline{\Delta}}{\overline{\Gamma} \Longrightarrow \overline{\Delta}, \neg\varphi} \quad \frac{\overline{\varphi}, \Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, \neg\overline{\varphi}}$$

$\wedge$:
$$\frac{\varphi, \Gamma \Longrightarrow \Delta}{\varphi \wedge \psi, \Gamma \Longrightarrow \Delta} \quad \frac{\psi, \Gamma \Longrightarrow \Delta}{\varphi \wedge \psi, \Gamma \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \overline{\Delta}, \varphi \quad \Gamma \Longrightarrow \overline{\Delta}, \psi}{\Gamma \Longrightarrow \overline{\Delta}, \varphi \wedge \psi}$$

$$\frac{\Gamma \Longrightarrow \Delta, \overline{\varphi} \quad \Gamma \Longrightarrow \Delta, \overline{\psi}}{\Gamma \Longrightarrow \Delta, \overline{\varphi} \wedge \overline{\psi}}$$

$\vee$:
$$\frac{\varphi, \overline{\Gamma} \Longrightarrow \Delta \quad \psi, \overline{\Gamma} \Longrightarrow \Delta}{\varphi \vee \psi, \overline{\Gamma} \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta, \varphi}{\Gamma \Longrightarrow \Delta, \varphi \vee \psi} \quad \frac{\Gamma \Longrightarrow \Delta, \psi}{\Gamma \Longrightarrow \Delta, \varphi \vee \psi}$$

$$\frac{\overline{\varphi}, \Gamma \Longrightarrow \Delta \quad \overline{\psi}, \Gamma \Longrightarrow \Delta}{\overline{\varphi} \vee \overline{\psi}, \Gamma \Longrightarrow \Delta}$$

$\rightarrow$:
$$\frac{\Gamma \Longrightarrow \overline{\Delta}, \varphi \quad \psi, \overline{\Pi} \Longrightarrow \Lambda}{(\varphi \rightarrow \psi), \Gamma, \overline{\Pi} \Longrightarrow \overline{\Delta}, \Lambda} \qquad \frac{\varphi, \overline{\Gamma} \Longrightarrow \overline{\Delta}, \psi}{\overline{\Gamma} \Longrightarrow \overline{\Delta}, (\varphi \rightarrow \psi)}$$

$\forall$:
$$\frac{\varphi(t), \Gamma \Longrightarrow \Delta}{\forall x\varphi(x), \Gamma \Longrightarrow \Delta}$$
where t is any term

$$\frac{\Gamma \Longrightarrow \overline{\Delta}, \varphi(a)}{\Gamma \Longrightarrow \overline{\Delta}, \forall x\varphi(x)} \quad \frac{\Gamma \Longrightarrow \Delta, \overline{\varphi}(a)}{\Gamma \Longrightarrow \Delta, \forall x\overline{\varphi}(x)}$$
where a is a free variable which does not occur in the lower sequent.

$\exists$:
$$\frac{\varphi(a), \overline{\Gamma} \Longrightarrow \Delta}{\exists x\varphi(x), \overline{\Gamma} \Longrightarrow \Delta} \quad \frac{\overline{\varphi}(a), \Gamma \Longrightarrow \Delta}{\exists x\overline{\varphi}(x), \Gamma \Longrightarrow \Delta}$$
where a is a free variable which does not occur in the lower sequent.

$$\frac{\Gamma \Longrightarrow \Delta, \varphi(t)}{\Gamma \Longrightarrow \Delta, \exists x\varphi(x)}$$
where t is any term

We use the following abbreviations :

$$\varphi \leftrightarrow \psi \overset{\text{def}}{\Longleftrightarrow} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$$
$$u \overset{\Box}{\in} v \overset{\text{def}}{\Longleftrightarrow} \Box(u \in v)$$

1.2.2. *Nonlogical axioms.* Lattice valued set theory **LZFZ** is is a theory based on the lattice valued logic Łwith the nonlogical axioms GA1–GA11 which were valid on lattice valued universes :

GA1. Equality: $\forall u \forall v\,(u=v \wedge \varphi(u) \to \varphi(v))$.
GA2. Extensionality: $\forall u, v\,(\forall x(x \in u \leftrightarrow x \in v) \to u=v)$.
GA3. Pairing: $\forall u, v \exists z\,(\forall x(x \in z \leftrightarrow (x=u \vee x=v)))$.
The set z satisfying $\forall x(x \in z \leftrightarrow (x=u \vee x=v))$ is denoted by $\{u, v\}$.
GA4. Union: $\forall u \exists z\,(\forall x(x \in z \leftrightarrow \exists y \in u(x \in y)))$.
The set z satisfying $\forall x(x \in z \leftrightarrow \exists y \in u(x \in y))$ is denoted by $\bigcup u$.
GA5. Power set: $\forall u \exists z\,(\forall x(x \in z \leftrightarrow x \subset u))$, where

$$x \subset u \overset{\text{def}}{\Longleftrightarrow} \forall y(y \in x \to y \in u).$$

The set z satisfying $\forall x(x \in z \leftrightarrow x \subset u)$ is denoted by $\mathcal{P}(u)$.
GA6. Infinity: $\exists u\,(\exists x(x \in u) \wedge \forall x(x \in u \to \exists y \in u(x \in y)))$.
GA7. Separation: $\forall u \exists v\,(\forall x(x \in v \leftrightarrow x \in u \wedge \varphi(x)))$.
The set v satisfying $\forall x(x \in v \leftrightarrow x \in u \wedge \varphi(x))$ is denoted by $\{x \in u \mid \varphi(x)\}$.
GA8. Collection:

$$\forall u \exists v \left(\forall x(x \in u \to \exists y \varphi(x, y)) \to \forall x(x \in u \to \exists y \overset{\square}{\in} v \varphi(x, y))\right).$$

GA9. $\in$-induction: $\forall x\,(\forall y(y \in x \to \varphi(y)) \to \varphi(x)) \to \forall x \varphi(x)$.
GA10. Zorn: $\mathrm{Gl}(u) \wedge \forall v\,(\mathrm{Chain}(v, u) \to \bigcup v \in u) \to \exists z\,\mathrm{Max}(z, u)$, where

$$\begin{aligned}
\mathrm{Gl}(u) &\overset{\text{def}}{\Longleftrightarrow} \forall x(x \in u \to x \overset{\square}{\in} u), \\
\mathrm{Chain}(v, u) &\overset{\text{def}}{\Longleftrightarrow} v \subset u \wedge \forall x, y(x, y \in v \to x \subset y \vee y \subset x), \\
\mathrm{Max}(z, u) &\overset{\text{def}}{\Longleftrightarrow} z \in u \wedge \forall x(x \in u \wedge z \subset x \to z = x).
\end{aligned}$$

GA11. Axiom of $\Diamond$: $\forall u \exists z \forall t(t \in z \leftrightarrow \Diamond(t \in u))$.
The set z satisfying $\forall t(t \in z \leftrightarrow \Diamond(t \in u))$ is denoted by $\Diamond u$.

We say that a formula φ is *global*, if $(\varphi \to \Box \varphi)$, and a set u is *global* ($\mathrm{Gl}(u)$), if $x \in u$ is global for all x.

1.2.3. *Well-Founded Relations in* **LZFZ**.

Any formula with two free variables determines a binary relation. For a binary relation $A(x, y)$, we use the following abbreviations:

$$x \in \mathrm{Dom}\, A \overset{\text{def}}{\Longleftrightarrow} \exists y A(x, y), \quad x \in \mathrm{Rge}\, A \overset{\text{def}}{\Longleftrightarrow} \exists y A(y, x),$$

$$x \in \mathrm{Fld}\, A \overset{\text{def}}{\Longleftrightarrow} \exists y(A(x, y) \vee A(y, x)).$$

A binary relation $\prec$ is said to be *well-founded* if the following conditions are satisfied:

WF1: $\forall x, y \neg(x \prec y \wedge y \prec x)$
WF2: $\forall x[x \in \mathrm{Fld}(\prec) \wedge \forall y(y \prec x \to \varphi(y)) \to \varphi(x)] \to \forall x(x \in \mathrm{Fld}(\prec) \to \varphi(x))$
WF3: $\forall x \exists y \forall z(z \prec x \to z \in y)$

In view of the axiom GA9 ($\in$-induction), it is clear that the relation $\in$ is itself a well-founded relation, and so is $\overset{\Box}{\in}$.

Singlton $\{x\}$ and ordered pair $\langle x,y\rangle$ are defined as usual:

$$\{x\} \overset{\text{def}}{=} \{x,x\}, \quad \langle x,y\rangle \overset{\text{def}}{=} \{\{x\},\{x,y\}\}$$

so that $x \in \{y\} \iff x=y$ and $\langle x,y\rangle = \langle x',y'\rangle \iff x=x' \wedge y=y'$ hold.

We say a binary relation $F(x,y)$ is *global*, if $\forall x,y(F(x,y) \to \Box F(x,y))$; and a global relation $F(x,y)$ is *functional*, if

$$\forall x,y,y'(F(x,y) \wedge F(x,y') \to y=y').$$

For a global functional relation F, we write $F(x)=y$ instead for $F(x,y)$. If F is a global functional relation and $\prec$ is a well-founded relation, then $\{\langle x,y\rangle \mid F(x,y) \wedge \Diamond(x \prec u)\}$ is denoted by $F_{\prec u}$, for each set $u \in \mathrm{Fld}(\prec)$. $F_{\prec u}$ is a set by WF3, GA11($\Diamond$) and GA8(Collection). Note that

$$(y \prec x \to \Box\varphi(y)) \iff (\Diamond(y \prec x) \to \Box\varphi(y)).$$

Theorem 1.1 (Recursion Principle). *Let $\prec$ be a well founded relation and H be a global functional relation such that $\forall x \exists y H(x,y)$. Then there exists a unique global functional relation F such that*

$$\mathrm{Dom}\, F = \mathrm{Fld}(\prec) \ \wedge\ \forall x\,(x \in \mathrm{Fld}(\prec) \to (F(x) = H(F_{\prec x}))).$$

Definition 1.1. We define the formula $\mathrm{Ord}(\alpha)$ ("α is an ordinal") in **LZFZ** as follows:

$$\begin{aligned} \mathrm{Tr}(\alpha) &\overset{\text{def}}{\iff} \forall \beta,\gamma(\beta \in \alpha \wedge \gamma \in \beta \to \gamma \in \alpha), \\ \mathrm{Ord}(\alpha) &\overset{\text{def}}{\iff} \mathrm{Gl}(\alpha) \wedge \mathrm{Tr}(\alpha) \wedge \forall \beta(\beta \in \alpha \to \mathrm{Gl}(\beta) \wedge \mathrm{Tr}(\beta)), \end{aligned}$$

where $\mathrm{Gl}(\alpha) \overset{\text{def}}{\iff} \forall \beta(\beta \in \alpha \to \beta \overset{\Box}{\in} \alpha)$.

As an immediate consequence of the above definition, we have:

Lemma 1.2.

(1) $\mathrm{Ord}(\alpha) \wedge \beta \in \alpha \to \mathrm{Ord}(\beta)$

(2) $\mathrm{Gl}(X) \wedge \forall x(x \in X \to \mathrm{Ord}(x)) \to \mathrm{Ord}(\bigcup X)$

Definition 1.2. A global well founded relation $\prec$ is called a *well-ordering* on a set u if

$$(\mathrm{Fld}(\prec) = u) \wedge (\prec \text{ is transitive}) \wedge (\prec \text{ is extensional}),$$

where

$$\begin{aligned} \prec \text{ is transitive} &\overset{\text{def}}{\iff} \forall x,y,z[(x \prec y) \wedge (y \prec z) \to (x \prec z)] \\ \prec \text{ is extensional} &\overset{\text{def}}{\iff} \forall x,y[x,y \in u \wedge \forall z(z \prec x \leftrightarrow z \prec y) \to x = y]. \end{aligned}$$

Theorem 1.3. *Every global set can be well-ordered, i.e. for every global set u, there exists a global well-ordering relation $\prec$ on u.*

Theorem 1.4. *If u is a global set and $\prec$ is a global well-ordering relation on u, then $\langle u, \prec\rangle$ is isomorphic to an ordinal $\langle \alpha, \in\rangle$, i.e. there exists ρ such that*

$$\begin{aligned}&(\rho : u \to \alpha) \wedge \rho(u) = \alpha \wedge \\ &\quad \forall x, y \in u\, (x \prec y \leftrightarrow \rho(x) \in \rho(y)) \wedge (x = y \leftrightarrow \rho(x) = \rho(y))).\end{aligned}$$

1.2.4. *Check sets.*

We define the notion of check set in **LZFZ** , by $\overset{\square}{\in}$-recursion:

$$\mathrm{ck}(x) \overset{\text{def}}{\iff} \forall t \left(t \in x \leftrightarrow t \overset{\square}{\in} x \wedge \mathrm{ck}(t)\right).$$

The class of check sets will be denoted by W, i.e.

$$x \in W \overset{\text{def}}{\iff} \mathrm{ck}(x).$$

On the lattice valued universe $V^{\mathcal{L}}$, "$[\![\mathrm{ck}(x)]\!] = 1$" means that x is of the form $\check{u}$. That is, $[\![x = \check{u}]\!] = 1$ for some set u, where $\check{u}$ is defined for any set u by

$$\mathcal{D}\check{u} = \{\check{x} \mid x \in u\} \qquad \check{u}(\check{x}) = 1 \quad \text{for } x \in u.$$

Theorem 1.5. *The following (1)–(9) are provable in* **LZFZ** *, for any formula φ.*

(1) $\forall^W x, y (x \in y \to x \overset{\square}{\in} y)$

(2) $\forall^W x_1 \cdots x_n [\varphi^W(x_1, \cdots, x_n) \to \square\varphi^W(x_1, \cdots, x_n)]$

(3) $\forall^W x (\forall^W y (y \in x \to \varphi^W(y)) \to \varphi^W(x)) \to \forall^W x \varphi^W(x)$

(4) $\forall \alpha [\mathrm{Ord}(\alpha) \leftrightarrow \mathrm{ck}(\alpha) \wedge \mathrm{Ord}^W(\alpha)]$

(5) $\mathrm{ck}(\emptyset)$, *where $\emptyset$ is the empty set.*

(6) $\forall^W x, y [\mathrm{ck}(\{x, y\}) \wedge \mathrm{ck}(\bigcup x) \wedge \mathrm{ck}(\{z \in x \mid \square\varphi(z)\})]$

(7) *The set of natural numbers ω is defined as follows:*

$$\begin{aligned}\mathrm{Suc}(y) &\overset{\text{def}}{\iff} (y = \emptyset \vee \exists z (y = z + 1)), \text{ where } z + 1 = z \cup \{z\},\\ \mathrm{HSuc}(y) &\overset{\text{def}}{\iff} (\mathrm{Suc}(y) \wedge \forall z (z \in y \to \mathrm{Suc}(z))), \quad \text{and}\\ \omega &\overset{\text{def}}{=} \{y : \mathrm{HSuc}(y)\}.\end{aligned}$$

Then $\mathrm{Ord}(\omega) \wedge \forall^W n \in \omega (n = \emptyset \vee \exists^W m \in n (n = m + 1))$.

(8) *If u is a global set, then there exists an ordinal $\alpha\in\mathrm{On}$ with a bijection $\rho:u\to\alpha$, where $\alpha\in\mathrm{On}\overset{\mathrm{def}}{\Longleftrightarrow}\mathrm{Ord}(\alpha)$, i.e.*

$$\exists^W\alpha\in\mathrm{On}\,\exists\rho[\rho:u\to\alpha\wedge\rho(u)=\alpha\wedge\forall x,y(x,y\in u\wedge\rho(x)=\rho(y)\to x=y].$$

An interpretation of **ZFC** in **LZFZ** is obtained by relativizing the range of quantifiers to check sets. Namely "the class W of check sets is a model of **ZFC** " is provable in **LZFZ** .

We denote quantifiers relativized on check sets by $\forall^W,\exists^W$, i.e.

$$\forall^W x\varphi(x)\overset{\mathrm{def}}{\Longleftrightarrow}\forall x(\mathrm{ck}(x)\to\varphi(x))$$

$$\exists^W x\varphi(x)\overset{\mathrm{def}}{\Longleftrightarrow}\exists x(\mathrm{ck}(x)\wedge\varphi(x)).$$

For a formula φ of **LZFZ** , φ^W is the formula obtained from φ by replacing all quantifiers $\forall x$, $\exists x$, by $\forall^W x$, $\exists^W x$, respectively.

Theorem 1.6 (Interpretation of ZFC). *If φ is a theorem of* **ZFC** *, then φ^W is provable in* **LZFZ** *. i.e. For a formula $\varphi(x_1,\cdots,x_n)$ of* **ZFC** *,*

$$\forall^W x_1,\cdots,x_n(\varphi^W\vee\neg\varphi^W)$$

is provable in **LZFZ** *, and for each nonlogical axiom A of* **ZFC** *, A^W is provable in* **LZFZ** *.*

The power set $\mathcal{P}(1)$ of 1 ($=\{\emptyset\}$) is a global set and a complete lattice with respect to the inclusion $\subset$. We write $\leqslant$ instead of $\subset$. Then $(\mathcal{P}(1),\leqslant)$ is a comlpete lattice. Let

$$(p\to q)=\{x\in 1\mid 0\in p\to 0\in q\},\quad \neg p=\{x\in 1\mid\neg(0\in p)\}.$$

$\to$ is the basic implication and $\neg$ is the corresponding negation on $\mathcal{P}(1)$.

For a sentence φ, let

$$|\varphi|\overset{\mathrm{def}}{=}\{t\in 1\mid\varphi\}.$$

$|\varphi|$ is an element of $\mathcal{P}(1)$, and $\varphi\Longleftrightarrow 0\in|\varphi|$. Thus, the complete lattice $\mathcal{P}(1)$ represents the truth value set of **LZFZ** .

The relation $\prec$ defined by

$$\alpha\prec\beta\overset{\mathrm{def}}{\Longleftrightarrow}\alpha,\beta\in\mathrm{On}\;\wedge\;\alpha\in\beta$$

is a well founded relation and $\mathrm{Fld}(\prec)=\mathrm{On}$. Thus, the induction on $\alpha\in\mathrm{On}$ is justified in **LZFZ** . Now we construct the $\mathcal{P}(1)$-valued sheaf model by induction on $\alpha\in\mathrm{On}$ as follows:

$$\begin{aligned}W_\alpha^{\mathcal{P}(1)}&=\{u\mid\exists\beta\in\alpha\exists\mathcal{D}u\subset W_\beta^{\mathcal{P}(1)}(\mathrm{Gl}(\mathcal{D}u)\wedge u:\mathcal{D}u\to\mathcal{P}(1))\}\\ W^{\mathcal{P}(1)}&=\bigcup_{\alpha\in\mathrm{On}}W_\alpha^{\mathcal{P}(1)}\end{aligned}$$

On $W^{\mathcal{P}(1)}$, the atomic relation $=$ and $\in$ are interpreted as

$$\begin{aligned}[\![x=y]\!] &= \bigwedge_{t\in\mathcal{D}x}(x(t)\to[\![t\in y]\!])\wedge\bigwedge_{t\in\mathcal{D}y}(y(t)\to[\![t\in x]\!])\\ [\![x\in y]\!] &= \bigvee_{t\in\mathcal{D}y}[\![x=t]\!]\wedge y(t).\end{aligned}$$

Logical operations $\wedge$, $\vee$, $\to$, $\neg$, $\forall$, $\exists$ are interpreted as the correspondent operations on $\mathcal{P}(1)$. Then every sentence on $W^{\mathcal{P}(1)}$ has its truth value in $\mathcal{P}(1)$, and we have

Theorem 1.7. *For every sentence φ, "$(0\in[\![\varphi]\!])\longleftrightarrow\varphi$" is provable in* **LZFZ** .

2. Implications

Let $\to_*$ be any implication defined in the language of **LZFZ**. That is, let $\to_*$ be a logical operator such that the following sequents hold.

$$\begin{aligned}\Box(\varphi\to_*\psi) &\iff \varphi\to\psi\\ \varphi\wedge(\varphi\to_*\psi) &\implies \psi\end{aligned}$$

We define the corresponding $=_*$ and $\in_*$ by induction:

$$\begin{aligned}u=_*v &\overset{\text{def}}{\iff} \forall x(x\in u\to_* x\in_* v)\wedge\forall x(x\in v\to_* x\in_* u)\\ u\in_* v &\overset{\text{def}}{\iff} \exists x(x\in v\wedge u=_* x).\end{aligned}$$

It is obvious that $=$ and $\in$ coincide with $=_*$ and $\in_*$, respectively, if u, v are check sets. Namely, we have

Theorem 2.1. *If $\to_*$ is an implication, then*

$$\mathrm{ck}(u)\wedge\mathrm{ck}(v)\implies(u=v\leftrightarrow u=_*v)\wedge(u\in v\leftrightarrow u\in_* v).$$

Let φ be a formula constructed from atomic formulas of the forms $u=v$ and $u\in v$ by using logical operations $\wedge$, $\vee$, $\to$, $\neg$, $\forall$, $\exists$. Then we define φ^* as follows. $(u=v)^*$ is $u=_*v$; $(u\in v)^*$ is $u\in_* v$; $(\varphi\wedge\psi)^*$ is $\varphi^*\wedge\psi^*$; $(\varphi\vee\psi)^*$ is $\varphi^*\vee\psi^*$; $(\varphi\to\psi)^*$ is $\varphi^*\to_*\psi^*$; $(\neg\varphi)^*$ is $\neg^*\varphi^*$, where $\neg^*$ is the negation corresponding to $\to_*$; $(\forall x\in u\varphi(x))^*$ is $\forall x\in u\varphi^*(x)$; $(\exists x\in u\varphi(x))^*$ is $\exists x\in u\varphi^*(x)$, where $\forall x\in u\varphi(x)$ is an abbreviation of $\forall x(x\in u\to\varphi)$ and $\exists x\in u\varphi(x)$ is an abbreviation of $\exists x(x\in u\wedge\varphi)$.

Corollary 2.2. *If $\to_*$ is an implication, and if $P(x_1,\cdots,x_n)$ is a formula with free variables $x_1,\cdots,x_n$, in which all quantifiers are bounded by check sets (that is, of the form $\forall x\in u\varphi(x)$ or the form $\exists x\in u\varphi(x)$ for a check set u), then*

$$\forall x_1,\cdots,x_n\,(\mathrm{ck}(x_1)\wedge\cdots\mathrm{ck}(x_n)\to\Box P(x_1,\cdots,x_n)\wedge(P(x_1,\cdots,x_n)\leftrightarrow P^*(x_1,\cdots,x_n)).$$

Hence, for any sets $u_1, \cdots u_n$ in the external universe V, $[\![P(\check{u}_1, \cdots, \check{u}_n)]\!]$ is global, and

$$[\![P(\check{u}_1, \cdots, \check{u}_n)]\!] = [\![P^*(\check{u}_1, \cdots, \check{u}_n)]\!] = \begin{cases} 1, & \text{if } P(u_1, \cdots, u_n) \\ 0, & \text{if } \neg P(u_1, \cdots, u_n). \end{cases}$$

Proof. By induction on the complexity of P. $\square$

2.1. Extra axiom and intuitionistic implication. The power set $\mathcal{P}(1)$ of 1 ($= \{\emptyset\}$) is a global set ordered by the inclusion $\subset$. We write $\leqslant$ instead of $\subset$. Then $(\mathcal{P}(1), \leqslant)$ is a comlpete lattice which represents the truth value set of **LZFZ** :

Let

$$(p \to q) = \{x \in 1 \mid 0 \in p \to 0 \in q\}, \quad \neg p = \{x \in 1 \mid \neg(0 \in p)\}.$$

$\to$ is the basic implication and $\neg$ is the corresponding negation on $\mathcal{P}(1)$.

For a sentence φ, let

$$|\varphi| \stackrel{\text{def}}{=} \{t \in 1 \mid \varphi\}.$$

$|\varphi|$ is an element of $\mathcal{P}(1)$, and $\varphi \iff 0 \in |\varphi|$. Then the logical operations are interpreted as operations on $\mathcal{P}(1)$. Thus, the complete lattice $\mathcal{P}(1)$ represents the truth value set.

If we assume "$\mathcal{P}(1)$ is a cHa", i.e."$\mathcal{P}(1)$ is distributive", in **LZFZ**, then we have the distributive law of the logic:

$$\varphi \wedge \exists x \psi(x) \iff \exists x(\varphi \wedge \psi(x))$$

In fact:

$$\begin{aligned} \varphi \wedge \exists x \psi(x) &\iff 0 \in |\varphi \wedge \exists x \psi(x)| \\ &\iff 0 \in |\varphi| \wedge \bigvee_x |\psi(x)| \\ &\iff 0 \in \bigvee_x |\varphi \wedge \psi(x)| \\ &\iff 0 \in |\exists x(\varphi \wedge \psi(x))| \\ &\iff \exists x(\varphi \wedge \psi(x)) \end{aligned}$$

Let the operator $\to_{\mathrm{I}}$ be defined by

$$(\varphi \to_{\mathrm{I}} \psi) \stackrel{\text{def}}{\iff} 0 \in \bigcup \{u \in \mathcal{P}(1) \mid \varphi \wedge (0 \in u) \to \psi\}.$$

$\to_{\mathrm{I}}$ is the intuitionistic implication. The corresponding $=_{\mathrm{I}}$ and $\in_{\mathrm{I}}$ are defined by induction in **LZFZ** :

$$\begin{aligned} u =_{\mathrm{I}} v &\stackrel{\text{def}}{\iff} \forall x(x \in u \to_{\mathrm{I}} x \in_{\mathrm{I}} v) \wedge \forall x(x \in v \to_{\mathrm{I}} x \in_{\mathrm{I}} u) \\ u \in_{\mathrm{I}} v &\stackrel{\text{def}}{\iff} \exists x(x \in v \wedge u =_{\mathrm{I}} x). \end{aligned}$$

Then we have:

Theorem 2.3. *It is provable in (LZFZ + "$\mathcal{P}(1)$ is a cHa") that the set theory with $\to_{\mathrm{I}}$, $=_{\mathrm{I}}$, $\in_{\mathrm{I}}$, as its implication, equality, and membership relation, is an intuitionistic set theory. That is,*

$$\varphi \wedge \exists x \psi(x) \Longleftrightarrow \exists x(\varphi \wedge \psi(x))$$

and axioms of the intuitionistic set theory are provable in (LZFZ + "$\mathcal{P}(1)$ is a cHa").

Proof. For each axiom φ of intuitionistic set theory, $[\![\varphi]\!] = 1$ on $W^{\mathcal{P}(1)}$. cf. [7] □

3. Numbers in LZFZ

The set ω of all natural numbers is constructed from 0 by the successor function $x \mapsto x+1$, where 0 is the empty set and $x+1 = x \cup \{x\}$. The integers are constructed as equivalence classes of pairs of natural numbers, the rational numbers are constructed as equivalence classes of pairs of integers, and finally, the real numbers are constructed by Dedekind's cuts of rational numbers. We denote the set of all integers by $\mathbb{Z}$, the set of all rational numbers by $\mathbb{Q}$, and the set of all real numbers by $\mathbb{R}$.

3.1. Equivalence relation in LZFZ.

Definition 3.1. A global relation $\sim$ is an *equivalence relation* on a set G if

(1) $\forall a, b(a \sim b \to a \in G \wedge b \in G)$
(2) $\forall a \in G(a \sim a)$
(3) $\forall a, b \in G(a \sim b \to b \sim a)$
(4) $\forall a, b, c \in G(a \sim b \wedge b \sim c \to a \sim c)$.

If $\sim$ is an equivalence relation on G, we use the following usual notations.

$$\begin{aligned} [a] &= \{b \in G \mid a \sim b\} \text{ for } a \in G, \\ G/\!\sim &= \{[a] \mid a \in G\}. \end{aligned}$$

Definition 3.2. For elements u, v of a set G, the pair $\langle u, v \rangle$ of u, v is defined by

$$\langle u, v \rangle \stackrel{\text{def}}{=} \{\{u\}, \{u, v\}\},$$

and the set of all pairs $\langle u, v \rangle$ of $u, v \in G$ is denoted by $G \times G$.

$$G \times G \stackrel{\text{def}}{=} \{\langle u, v \rangle \mid u \in G \wedge v \in G\}.$$

Since $\mathrm{ck}(x_1) \wedge \cdots \wedge \mathrm{ck}(x_n)$ implies $\mathrm{ck}(\{x_1, \cdots, x_n\})$, we have

$$\mathrm{ck}(u) \wedge \mathrm{ck}(v) \to \mathrm{ck}(\langle u, v \rangle) \text{ and } \mathrm{ck}(G) \to \mathrm{ck}(G \times G).$$

Theorem 3.1. *Let G be a check set and let $P(x_1, x_2)$ be a formula with free variables x_1, x_2, in which all quantifiers are bounded by check sets. If $P(x_1, x_2)$ defines an equivalence relation $\sim$ on G, i.e. if the relation $\sim$ defined by*

$$a \sim b \longleftrightarrow a \in G \wedge b \in G \wedge P(a, b)$$

is an equivalence relation on G, then the relation $\sim$ is also a check set, and

$$\mathrm{ck}(\sim),\ \mathrm{ck}([a]) \textit{ for } a \in G, \quad \textit{and} \quad \mathrm{ck}(G/\sim).$$

Proof. Since $\mathrm{ck}(G \times G)$ and $\mathrm{Gl}(P(a, b))$, the relation $\sim$ defined by

$$\sim = \{\langle a, b\rangle \in G \times G \mid P(a, b)\}$$

is a check set. It follows that $\mathrm{ck}([a])$ for $a \in G$, and $\mathrm{ck}(G/\sim)$ by Corollary 2.2. □

3.2. Natural numbers in $V^{\mathcal{L}}$.

The sentence "x is a natural number" is defined as follows :

$$\begin{aligned} \mathrm{Suc}(x) &\overset{\mathrm{def}}{\iff} x = \check{0} \vee \exists y(x = y + 1), \\ \mathrm{HSuc}(x) &\overset{\mathrm{def}}{\iff} \mathrm{Suc}(x) \wedge \forall y(y \in x \to \mathrm{Suc}(y)), \\ \text{"}x \text{ is a natural number"} &\overset{\mathrm{def}}{\iff} \mathrm{HSuc}(x). \end{aligned}$$

$\{x \mid \mathrm{HSuc}(x)\}$ is a check set, which is equal to $\check{\omega}$ in the universe $V^{\mathcal{L}}$. So we denote the set of natural numbers $\{x \mid \mathrm{HSuc}(x)\}$ by $\check{\omega}$.

We define the addition $+$ and multiplication $\cdot$ on $\check{\omega}$, as usual. The check sets $\check{+}$, and $\check{\cdot}$ associated with the operations $+$, $\cdot$ on ω coincide with those in $V^{\mathcal{L}}$. That is, let

$$\begin{cases} \mathcal{D}(+) = \{\langle \check{m}, \check{n}, (m+n)\check{}\,\rangle \mid m, n \in \omega\} \\ +(\check{m}, \check{n}, (m+n)\check{}\,) = 1 \end{cases}$$

$$\begin{cases} \mathcal{D}(\cdot) = \{\langle \check{m}, \check{n}, (m \cdot n)\check{}\,\rangle \mid m, n \in \omega\} \\ \cdot\,(\check{m}, \check{n}, (m \cdot n)\check{}\,) = 1. \end{cases}$$

We denote $\langle x, y, z\rangle \in +$, and $\langle x, y, z\rangle \in \cdot$ by $x + y = z$, and $x \cdot y = z$, respectively. Then $+$, $\cdot$ are operations on $\check{\omega}$ in $V^{\mathcal{L}}$, and for $m, n \in \omega$,

$$[\![\check{m} + \check{n} = (m+n)\check{} \wedge \check{m} \cdot \check{n} = (m \cdot n)\check{}\,]\!] = 1.$$

Similarly, the relation associated with the relation $\leqslant$ on ω is defined and is also denoted by $\leqslant$. $\leqslant$ is the relation on $\check{\omega}$. That is, let

$$\begin{cases} \mathcal{D}(\leqslant) = \{\langle \check{m}, \check{n}\rangle \mid m, n \in \omega,\ m \leqslant n\} \\ \leqslant (\check{m}, \check{n}) = 1. \end{cases}$$

We denote $\leqslant (x,y)$ by $x \leqslant y$. Then, $m \leqslant n$ iff $[\![\check{m} \leqslant \check{n}]\!] = 1$ for all $m, n \in \omega$, and

$$\forall m, n[m, n \in \check{\omega} \rightarrow (m \leqslant n \leftrightarrow \exists l(l \in \check{\omega} \wedge m + l = n)].$$

It follows that if $\varphi(x_1, \cdots, x_n)$ is a bounded formula on $\check{\omega}$ which is constructed in terms of the relations $\in$, $=$, $\leqslant$ and functions $+$, $\cdot$, then for all $x_1, \cdots, x_n \in \omega$

$$\varphi(x_1, \cdots, x_n) \iff [\![\varphi(\check{x_1}, \cdots, \check{x_n})]\!] = 1.$$

3.3. Integers.

Integers are defined to be equivalence classes of $\omega \times \omega$, where the equivalence relation $\sim$ is defined by a bounded formula :

$$u \sim v \overset{\text{def}}{\iff} \exists m, n, p, q \in \omega(u = \langle m, n\rangle \wedge v = \langle p, q\rangle \wedge m + q = n + p).$$

That is, $\mathbb{Z} = \omega \times \omega / \sim$.

The corresponding equivalence relation $\sim$ on $\check{\omega} \times \check{\omega}$ is defined by

$$u \sim v \overset{\text{def}}{\iff} \exists m, n, p, g \in \check{\omega}(u = \langle m, n\rangle \wedge v = \langle p, g\rangle \wedge m + q = n + p),$$

and $\check{\omega} \times \check{\omega}/\sim$ is a check set. On $V^{\mathcal{L}}$, we have

$$[\![\check{\omega} \times \check{\omega}/\sim = (\omega \times \omega/\sim)\check{} = \check{\mathbb{Z}}]\!] = 1.$$

So we denote the set of all integers by $\check{\mathbb{Z}}$, and operations $+$ and $\cdot$ on $\check{\mathbb{Z}}$ are defined as usual. On $V^{\mathcal{L}}$, we have

$$[\![\check{a} + \check{b} = (a+b)\check{} \wedge \check{a} \cdot \check{b} = (a \cdot b)\check{}]\!] = 1,$$

$$[\![\check{a} \leqslant \check{b}]\!] = \begin{cases} 1 & \text{if } a \leqslant b \\ 0 & \text{if } a > b, \end{cases} \qquad [\![\check{a} < \check{b}]\!] = \neg[\![\check{a} \geqslant b]\!].$$

3.4. Rational numbers.

In order to define the set $\mathbb{Q}$ of rational numbers, we define an equivalence relation $\sim$ on $\mathbb{Z} \times \mathbb{Z}$ by a bounded formula :

$$u \sim v \overset{\text{def}}{\iff} \exists a, b, a', b' \in \mathbb{Z}(u = \langle a, b\rangle \wedge v = \langle a', b'\rangle \wedge ab' = a'b).$$

Then the set $\mathbb{Q}$ is defined to be $(\mathbb{Z} \times \mathbb{Z})/\sim$. The corresponding equivalence relation $\sim$ on $(\check{\mathbb{Z}} \times \check{\mathbb{Z}})$ is defined by

$$u \sim v \overset{\text{def}}{\iff} \exists a, b, a', b' \in \check{\mathbb{Z}}(u = \langle a, b\rangle \wedge v = \langle a', b'\rangle \wedge ab' = a'b).$$

Then $(\check{\mathbb{Z}} \times \check{\mathbb{Z}})/\sim$ is a check set such that

$$[\![(\check{\mathbb{Z}} \times \check{\mathbb{Z}})/\sim = \check{\mathbb{Q}}]\!] = 1$$

on $V^{\mathcal{L}}$. We denote the set of all rational numbers by $\check{\mathbb{Q}}$. Operators $+, \cdot$ and relations $\leqslant, <$ on $\check{\mathbb{Q}}$ are defined so that

$$[\![\check{a} + \check{b} = (a+b)^{\check{}} \wedge \check{a} \cdot \check{b} = (a \cdot b)^{\check{}}]\!] = 1$$

$$[\![\check{a} \leqslant \check{b}]\!] = \begin{cases} 1 & \text{if } a \leqslant b \\ 0 & \text{if } a > b \,, \end{cases} \qquad [\![\check{a} < \check{b}]\!] = \neg [\![\check{a} \geqslant \check{b}]\!]$$

on $V^{\mathcal{L}}$.

3.5. Real numbers.

By a real number we mean a Dedekind cut of rational numbers. That is,

Definition 3.3. A set $a = \langle L, U \rangle \in \mathcal{P}(\check{\mathbb{Q}}) \times \mathcal{P}(\check{\mathbb{Q}})$ is a *real number* if $P_1(L,U) \wedge \cdots \wedge P_6(L,U))$, where

$$\begin{aligned}
P_1(L,U) &: \exists r \in \check{\mathbb{Q}}(r \in L) \wedge \exists s \in \check{\mathbb{Q}}(s \in U) \\
P_2(L,U) &: \forall r \in \check{\mathbb{Q}}\,((r \in L) \longrightarrow \neg(r \in U)) \ ; \\
&\quad\ \forall r \in \check{\mathbb{Q}}((r \in U) \longrightarrow \neg(r \in L)) \\
P_3(L,U) &: \forall r \in \check{\mathbb{Q}}\,\Big((r \in L) \longleftrightarrow \exists s \in \check{\mathbb{Q}}(r < s \wedge s \in L)\Big) \\
P_4(L,U) &: \forall r \in \check{\mathbb{Q}}\,\Big((r \in U) \longleftrightarrow \exists s \in \check{\mathbb{Q}}(s < r \wedge s \in U)\Big) \\
P_5(L,U) &: \forall r, s \in \check{\mathbb{Q}}\,(s < r \longrightarrow s \in L \vee r \in U)) . \\
P_6(L,U) &: \forall r, s \in \check{\mathbb{Q}}\,(s < r \wedge \neg(s \in L) \longrightarrow r \in U)) .
\end{aligned}$$

$\mathcal{P}(\check{\mathbb{Q}})$ is not a check set. Therefore, we cannot use the same argument as we used for rational numbers to say "the set of real numbers is a check set". However, it is a fact because of the property of the basic implication and the corresponding negation, as we will see.

Theorem 3.2. (u *is a real number*) $\rightarrow \mathrm{ck}(u)$.

Proof. Let $u = \langle L, U \rangle$.

$$\begin{aligned}
P_2(L,U) \wedge \cdots \wedge P_6(L,U) \wedge (r \in L) &\longrightarrow r \in L \wedge \exists s \in \mathbb{Q}(r < s \wedge s \in L) \\
&\longrightarrow r \in L \wedge \exists s \in \mathbb{Q}(r < s \wedge \neg(s \in U)) \\
&\longrightarrow \Box(r \in L)
\end{aligned}$$

Since $P_2(L,U), \cdots, P_6(L,U)$ are all global,

$$P_2(L,U) \wedge \cdots \wedge P_6(L,U) \wedge (r \in L) \longrightarrow \Box(r \in L).$$

Therefore, L is a check set. Similarly, U is a check set. $\Box$

4. Global intuitionistic set theory

In intuitionistic logic, we have the logical distributive law:

$$\forall x(\varphi(x) \wedge \psi) \leftrightarrow \forall x\varphi(x) \wedge \psi.$$

The logical distributive law is equivalent to the sentence "$\mathcal{P}(1)$ is a **cHa** " in intuitionistic set theory.

In the set theory **LZFZ** with the axiom "$\mathcal{P}(1)$ is a **cHa** " added, we define the intuitionistic implication $\rightarrow_{\mathrm{I}}$, the intuitionistic equality $=_{\mathrm{I}}$, and the intuitionistic membership relation $\in_{\mathrm{I}}$, by using recursion principle, as follows:

$$\begin{aligned}
\varphi \rightarrow_{\mathrm{I}} \psi &\iff \exists x \in 1(\varphi \wedge (0 \in x) \rightarrow \psi) \\
u =_{\mathrm{I}} v &\iff \forall x(x \in u \rightarrow_{\mathrm{I}} x \in_{\mathrm{I}} v) \wedge \forall y(y \in v \rightarrow_{\mathrm{I}} y \in_{\mathrm{I}} u) \\
u \in_{\mathrm{I}} v &\iff \exists y \in v(u =_{\mathrm{I}} y)
\end{aligned}$$

then the inference rules and axioms of the intuitionistic set theory **IZFZ** with respect to $=_{\mathrm{I}}$, $\in_{\mathrm{I}}$, are provable in **LZFZ** + "$\mathcal{P}(1)$ is a **cHa** ", and the set theory **LZFZ** + "$\mathcal{P}(1)$ is a **cHa** " is equivalent to the global intuitionistic set theory **GIZFZ** in [7]. From now on, we denote the global intuitionistic set theory **LZFZ** + "$\mathcal{P}(1)$ is a **cHa** " by **GIZFZ** .

5. Sheaf structure of sets

Now we have two equalities $=$ and $=_{\mathrm{I}}$ in **GIZFZ** , and

$$u = v \rightarrow u =_{\mathrm{I}} v.$$

That is, $=_{\mathrm{I}}$ can be considered as an equivalence relation in **GIZFZ** .

First we review the definition of sheaf in the classical set theory **ZFC** .

Definition 5.1. Let Ω be a **cHa** . A triple $\langle A, E, \lceil \rangle$ of a set A and maps $E : A \rightarrow \Omega$ and $\lceil : A \times \Omega \rightarrow A$ is called a *presheaf over* Ω if for every $a, b \in A$ and $p, q \in \Omega$,
(1) $a\lceil 0 = b \lceil 0$
(2) $a \lceil Ea = a$
(3) $E(a\lceil p) = Ea \wedge p$
(4) $(a\lceil p)\lceil q = a\lceil(p \wedge q)$

If $\langle A, E, \lceil \rangle$ is a presheaf over Ω, and if a subset F of A satisfies

$$f\lceil Eg = g\lceil Ef \qquad \text{for all } g, f \in F,$$

then F is said to be *compatible*. A presheaf $\langle A, E, \lceil \rangle$ over Ω is called a *sheaf over* Ω if

(5) For every compatible subset F of A there exists a unique g in A such that

$$\begin{array}{ll} (5.1) & g\lceil Ef = f \quad \text{for } f \in F \\ (5.2) & Eg = \bigvee\{Ef \mid f \in F\}. \end{array}$$

Let Ω be a **cHa** and V^Ω be the Ω-valued sheaf model. On V^Ω, the relation $\sim$ given by

$$x \sim y \overset{\text{def}}{\Longleftrightarrow} [\![x =_{\mathrm{I}} y]\!] = 1$$

is an equivalence relation. Define $E : V^\Omega \to \Omega$ and $\lceil : V^\Omega \times \Omega \to V^\Omega$ by

$$Ex = [\![x \in_{\mathrm{I}} u]\!],$$
$$\begin{cases} \mathcal{D}(x\lceil p) = \{t\lceil p \mid t \in \mathcal{D}x\} \\ (x\lceil p)(t\lceil p) = \bigvee\{x(t') \wedge p \mid t' \in \mathcal{D}x \wedge (t\lceil p = t'\lceil p)\}. \end{cases}$$

Then it is known that for $x, y \in V^\Omega$ and $p, q \in \Omega$,
(1) $[\![x\lceil 0 =_I y\lceil 0]\!] = 1$
(2) $p \le [\![x\lceil p =_{\mathrm{I}} x]\!]$
(3) $[\![t \in_{\mathrm{I}} x\lceil p]\!] = [\![t \in_{\mathrm{I}} x]\!] \wedge p$
(4) $E(x\lceil p) = Ex \wedge p$
(5) $[\![(x\lceil p)\lceil q =_{\mathrm{I}} x\lceil p \wedge q]\!] = 1$

Since $[\![t =_{\mathrm{I}} y]\!] = 1$ implies $Ex = Ey$ and $[\![x\lceil p =_{\mathrm{I}} y\lceil p]\!] = 1$, E, $\lceil$ induce the operations on the quotient $V^\Omega/_\sim$, i.e.

$$E : V^\Omega/_\sim \to \Omega; \quad \lceil : V^\Omega/_\sim \times \Omega \to V^\Omega/_\sim.$$

An element u of V^Ω is called a *sheaf representation* if $\langle \mathcal{D}u/_\sim, E, \lceil \rangle$ is a sheaf.

It is known that for every set $u \in V^\Omega$ satisfying $[\![\forall x \in_{\mathrm{I}} u \exists t (t \in_{\mathrm{I}} x)]\!] = 1$, there exists a sheaf representation $v \in V^\Omega$ such that $[\![u =_{\mathrm{I}} v]\!] = 1$.

Now recall that V^Ω is a standard model of **GIZFZ** , and Ω represents $\mathcal{P}(1)$ in the set theory **IZFZ** . That is, $\mathcal{P}(1)$ has the whole information of $\mathcal{L}$, where the order on $\mathcal{P}(1)$ is the inclusion $\subset$.

This enables us to define the structure of sheaf representation in **GIZFZ** .

5.1. Definition of sheaf structure. Now we return to the discussion in **GIZFZ.** Let $\mathcal{P}(1)$ be the power set of $1 = \{\emptyset\}$. $\mathcal{P}(1)$ is a complete Heyting algebra (*cHa*) with respect to the inclusion $\supset$. We denote $p \subset q$, $p \cup q$, $p \cap q$ on $\mathcal{P}(1)$, by $p \le q$, $p \vee q$, $p \wedge q$, respectively.

Definition 5.2. Let u be a given set. $\langle A, E, \lceil \rangle$ is called a *presheaf structure of* u if A is a global set (i.e. $\forall x (x \in A \Longrightarrow x \overset{\Box}{\in} A)$) and $E : A \to \mathcal{P}(1)$, $\lceil : A \times \mathcal{P}(1) \to A$ are function satisfying the following conditions, where $f : X \to Y$ (f is a function from X to Y) means $\mathrm{Gl}(f) \wedge \forall x (x \in X \Longrightarrow \exists! y \in Y (\langle x, y \rangle \in f))$:
(1) $\quad \forall x \in A (0 \in Ex \Longleftrightarrow x \in_{\mathrm{I}} u)$

(2) $\forall x \in A \forall p \in \mathcal{P}(1)(x\lceil p = \{t \in x \mid 0 \in p\})$
(3) $\forall x, y \in A(x\lceil 0 = y\lceil 0)$
(4) $\forall x \in A((x\lceil Ex) = x)$
(5) $\forall x \in A \forall p \in \mathcal{P}(1)(E(x\lceil p) = (Ex \wedge p))$
(6) $\forall x \in A \forall p, q \in \mathcal{P}(1)((x\lceil q)\lceil p = x\lceil (p \wedge q))$
(7) $u = \{x \in A \mid 0 \in Ex\}$.

Definition 5.3. Let $\langle A, E, \lceil\rangle$ be a presheaf structure of a set u. If a global subset F of A satisfies

$$f\lceil Eg = g\lceil Ef \qquad \text{for all } f, g \in F,$$

then F is said to be *compatible*, and written as Comp(F). i.e.

$$\mathrm{Comp}(F) \overset{\text{def}}{\Longleftrightarrow} \forall f, g \in F(f\lceil Eg = g\lceil Ef).$$

The presheaf structure $\langle A, E, \lceil\rangle$ of u is called a *sheaf structure of* u if

(8) For every global and compatible subset F of A there exists $g \in A$ such that

$$(8.1) \quad f \in F \Longrightarrow g\lceil Ef = f$$
$$(8.2) \quad Eg = \bigvee\{Ef \mid f \in F\}.$$

If E and $\lceil$ are obvious for a sheaf structure $\langle A, E, \lceil\rangle$, then we write simply A instead of $\langle A, E, \lceil\rangle$.

Theorem 5.1. *Every set u with $\forall x(x \in u \to \exists t(t \in x))$ has its sheaf structure.*

Proof. For $x \in \Diamond u$ and $p \in \mathcal{P}(1)$, let

$$Ex = \{t \in 1 \mid x \in_{\mathrm{I}} u\},$$
$$x\lceil p = \{t \in x \mid 0 \in p\},$$

and let

$$A' = \{x\lceil (Ex \wedge p) \mid x \in \Diamond u,\ p \in \mathcal{P}(1)\}.$$

Then we have

(1) $x \in A' \wedge p \in \mathcal{P}(1) \Longrightarrow x\lceil p \in A'$
(2) $x \in A' \Longrightarrow x\lceil Ex = x$
(3) $E(x\lceil p) = Ex \wedge p$
(4) $u = \{x \in A' \mid 0 \in Ex\}$
(5) If $(F \subset A') \wedge \mathrm{Gl}(F) \wedge \mathrm{Comp}(F)$, where $\mathrm{Gl}(F) \overset{\text{def}}{\Longleftrightarrow} \forall x(x \in F \Longrightarrow x \overset{\Box}{\in} F)$, and $\bigvee F \overset{\text{def}}{=} (\cup F)\lceil \bigvee\{Ea \mid a \in F\}$, then
  (i): $\forall f(f \in F \wedge 0 \in Ef \Longrightarrow f =_{\mathrm{I}} \bigvee F)$
  (ii): $E(\bigvee F) = \bigvee\{Ef \mid f \in F\}$
(6) $\mathrm{Gl}(F) \wedge \mathrm{Comp}(F) \Longrightarrow (\bigvee F)\lceil p = \bigvee\{f\lceil p \mid f \in F\}$.

Let $A = \{\bigvee F \mid F \subset A' \wedge \mathrm{Comp}(F) \wedge \mathrm{Gl}(F)\}$. $\langle A, E, \lceil\rangle$ is a sheaf structure of u. □

Theorem 5.2. *If* $\langle \tilde{S}, E, \lceil \rangle$ *is a sheaf structure of a set* S, *then*

$$\forall a \exists a' \in \tilde{S} \; (a \in_{\mathrm{I}} S \Longrightarrow a =_{\mathrm{I}} a')$$

Proof. Let

$$F = \left\{ b \lceil p_b \mid (b \in \tilde{S}) \wedge (p_b = (\{t \in 1 \mid b =_{\mathrm{I}} a\} \wedge Ea)) \right\}.$$

Comp(F), hence $\bigvee F \in \tilde{S}$ and $(a \in_{\mathrm{I}} S \to a =_{\mathrm{I}} \bigvee F)$. □

Now let X be a topological space and Ω be the **cHa** consisting of all open sets of X. Then Ω is a **cHa** and V^{Ω} is a model of **GIZFZ**. The definition of intuitionistic real numbers is obtained from the definition of real numbers by replacing $\to$, $=$ and $\in$ by $\to_{\mathrm{I}}$, $=_{\mathrm{I}}$ and $\in_{\mathrm{I}}$, respectively. Let R of all intuitionistic real numbers in **GIZFZ**. The sheaf structure $\tilde{R}$ of R of represents the sheaf of germs of continuous functions on X.

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